Digital Communication Systems
EES 452

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3 An Introduction to
Digital Communication Systems
Over Discrete Memoryless Channel

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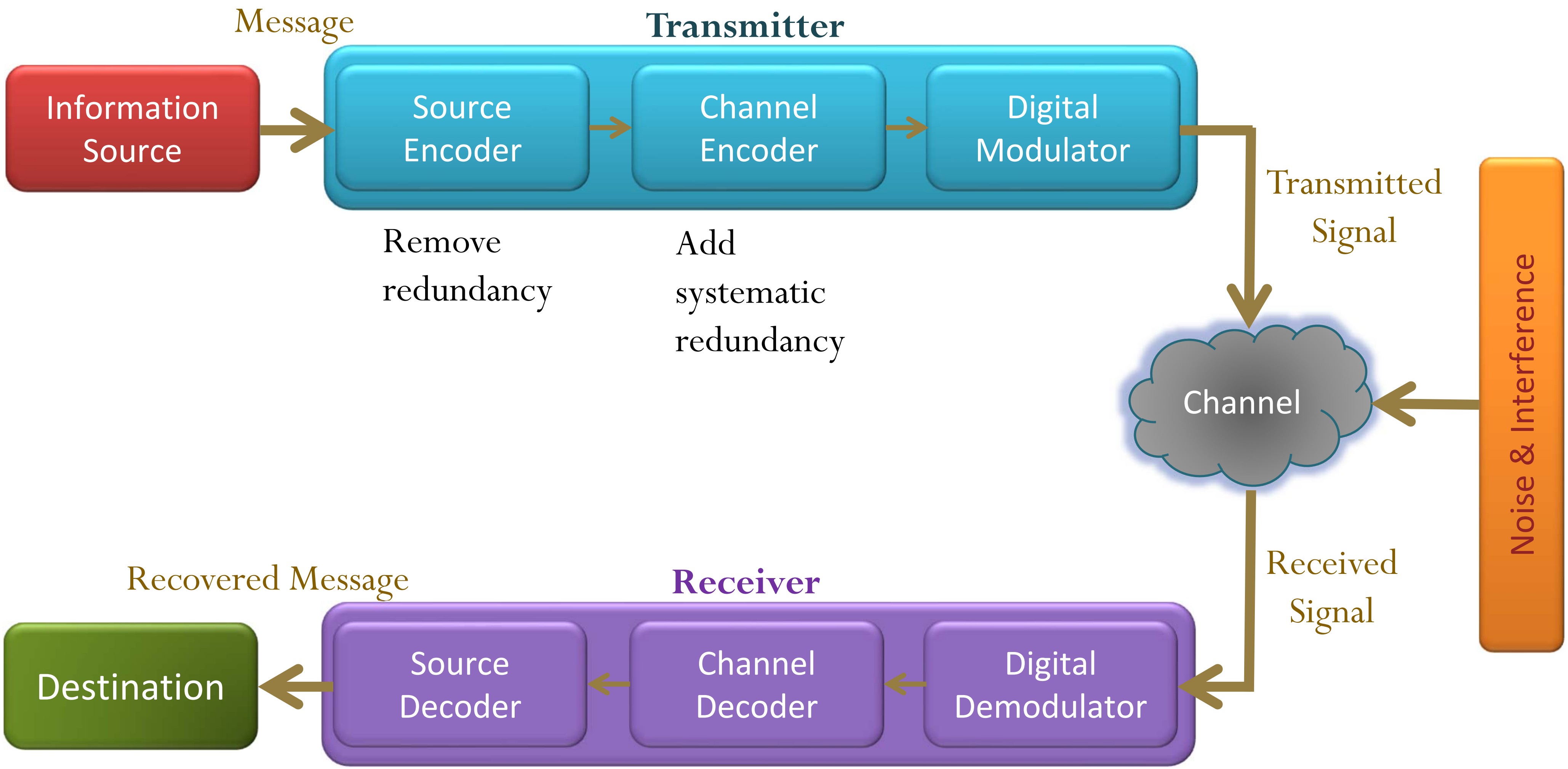
3 An Introduction to Digital Communication Systems Over Discrete Memoryless Channel

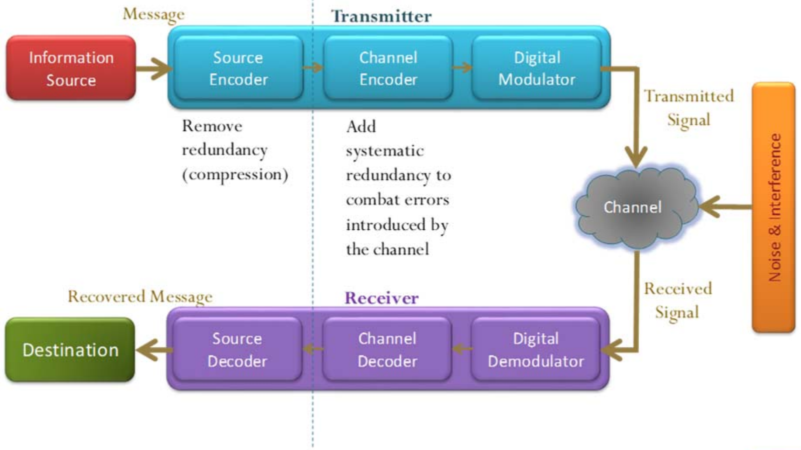
3.1 DMC Models:
Equivalent Channel; Binary Symmetric Channel

Elements of digital commu. sys.

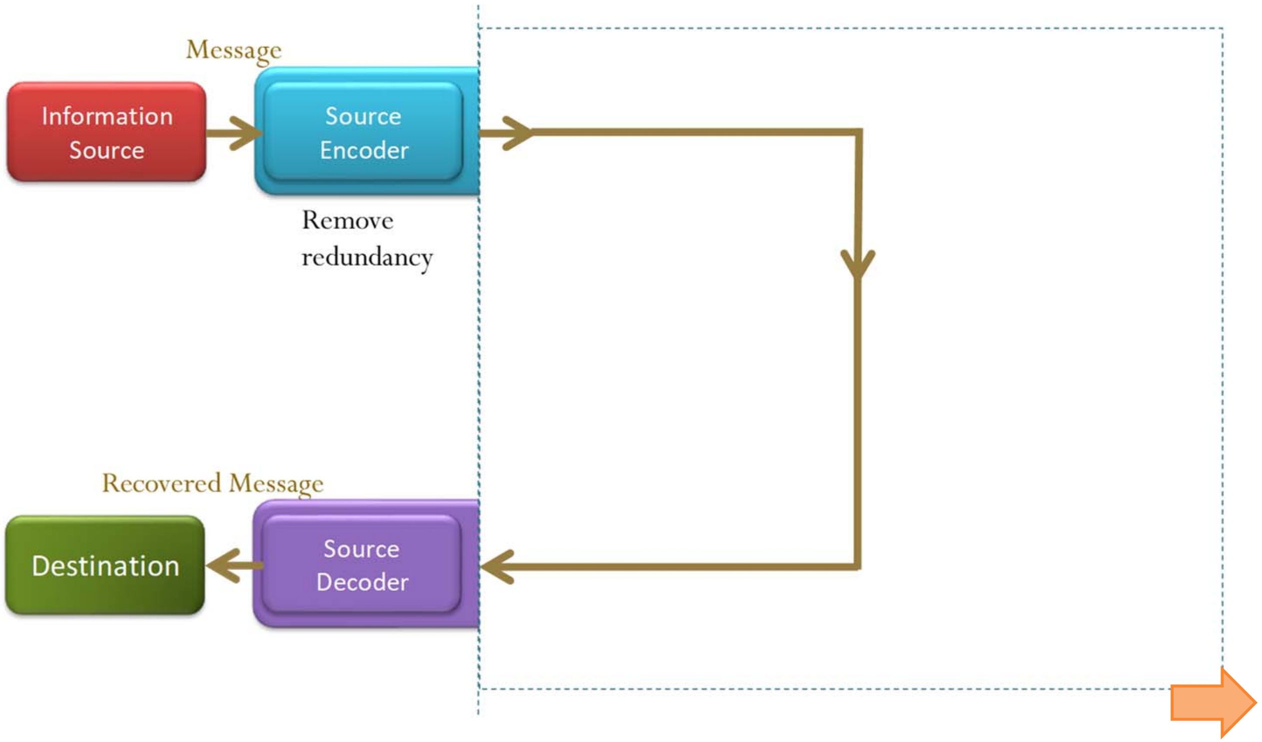


System Under Consideration

Digital communication system

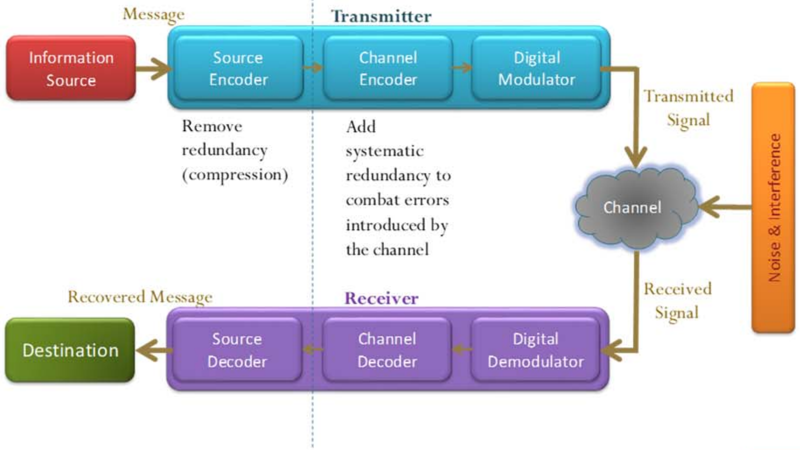


System considered back in CH2

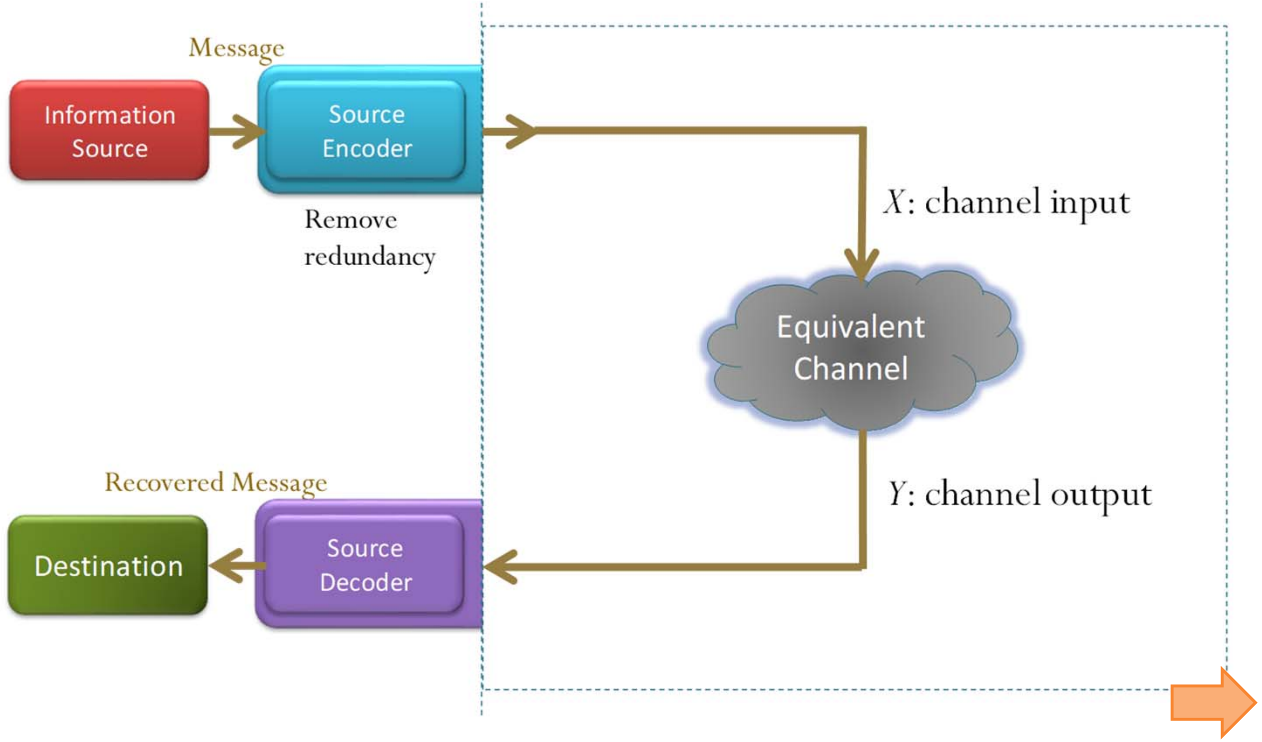


System Under Consideration

Digital communication system

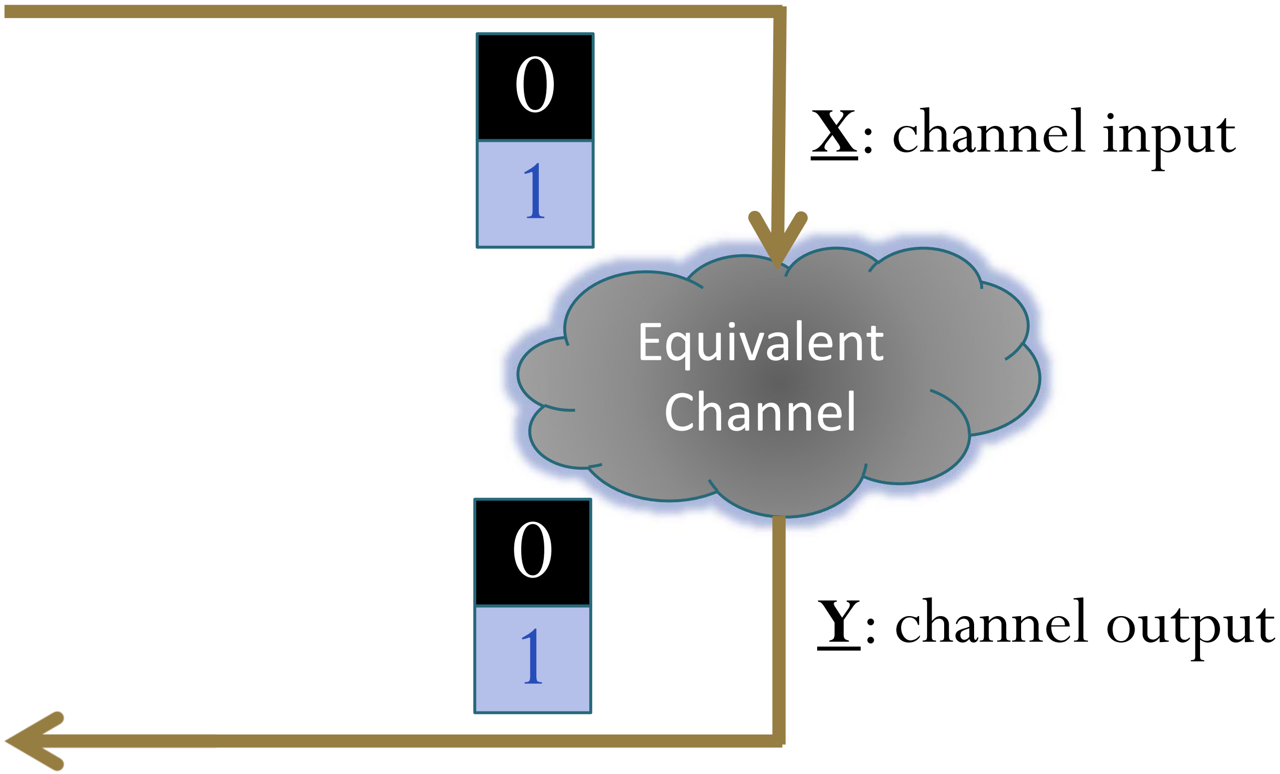


System considered in Section 3.1



Noisy Channel

111111111111111111111111111111111111111111111111111
111111111111111111111111111111000000000000111111…



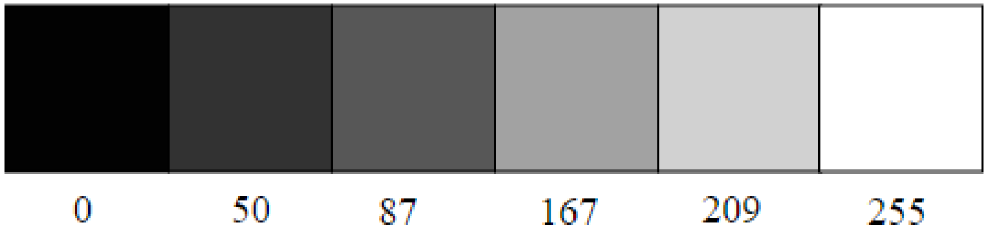
110110111101111011111011111110101001111011110111111
110011111111111001111111011111001000000000101101…

[BSC_demo_image.m]

MATLAB

```
x_raw = imresize(imread('SIIT_25LOGO.png'),[150 150]);
x_raw = [rgb2gray(x_raw) > 128]; % convert to 0 and 1
n = prod(size(x_raw));
x = reshape(x_raw,1,n); % convert to row vector

figure
x_img = reshape(x,size(x_raw));
imshow(x_img*255);
```



[BSC_demo_image.m]

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3 An Introduction to Digital Communication Systems Over Discrete Memoryless Channel

3.1 DMC Models:

Estimating Probability by Relative Frequency; Conditional and Joint Probability Mass Functions for Two Random Variables

Conditional pmf

Conditional probability

$$P(A|B) \equiv \frac{P(A \cap B)}{P(B)}$$

Conditional pmf

$$p_{Y|X}(y \mid x) \equiv P[Y = y \mid X = x]$$

$$\equiv P([Y = y] | [X = x])$$

$A = [Y = y]$

$B = [X = x]$

Two Random Variables

Joint probability

$$P\big(\underbrace{A\cap B}_{\text{Joint event}}\big)$$

Joint pmf

$$p_{X,Y}(x,y)\equiv P[X=x,Y=y]$$

"and"

$$\equiv P\big([X=x]\cap[Y=y]\big)$$

$$A=[Y=y]$$

$$B=[X=x]$$

Conditional probability

$$P(A|B)\equiv\frac{P(A\cap B)}{P(B)}$$

Conditional pmf

$$p_{Y|X}(y\,|\,x)\equiv P[Y=y\,|\,X=x]$$

$$\equiv P\big([Y=y]\,\big|\,[X=x]\big)$$

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3 An Introduction to Digital Communication Systems Over Discrete Memoryless Channel

3.1 DMC Models:

Channel Input X and Output Y, Their PMF p(x) and q(y), Alphabets, and Probability Vectors

MATHEMATICAL SCRIPT CAPITAL X

Tweet

Your Browser	𝒳
Decomposition	X U+0058
Index	U+1D4B3 (119987)
Class	Uppercase Letter (Lu)
Block	Mathematical Alphanumeric Symbols
Java Escape	"\ud835\udcb3"
Javascript Escape	"\ud835\udcb3"
Python Escape	u'\U0001d4b3'
HTML Escapes	𝒳 𝒳
URL Encoded	q=%F0%9D%92%B3
UTF8	f0 9d 92 b3
UTF16	d835 dcb3

https://charbase.com/1d4b3-unicode-mathematical-script-capital-x

MATHEMATICAL SCRIPT CAPITAL Y

Tweet

Your Browser	𝒴
Decomposition	Y U+0059
Index	U+1D4B4 (119988)
Class	Uppercase Letter (Lu)
Block	Mathematical Alphanumeric Symbols
Java Escape	"\ud835\udcb4"
Javascript Escape	"\ud835\udcb4"
Python Escape	u'\U0001d4b4'
HTML Escapes	𝒴 𝒴
URL Encoded	q=%F0%9D%92%B4
UTF8	f0 9d 92 b4
UTF16	d835 dcb4

https://charbase.com/1d4b4-unicode-mathematical-script-capital-y

ECS 315: In-Class Exercise # 7 - Sol

Instructions

1. Separate into groups of no more than three students each. **The group cannot be the same as any of your former groups.**
2. Explanation is **not required** for this exercise. [ENRE]
3. **Do not panic.**

Date: 0 5 / 0 9 / 2019			
Name	ID (last 3 digits)		
Prapun	5	5	5

1) Consider the following sequences of 1s and 0s which summarize the data obtained from 15 testees in a disease testing experiment.

```
D:  0 1 1 0 0 0 0 1 1 1 0 1 0 1 1
TP: 0 0 0 0 1 0 1 1 0 0 1 0 0 1 1
```

The results in the i-th column are for the i-th testee. The D row indicates whether each of the testees actually has the disease under investigation. The TP row indicates ... Numbers "1" and "0" correspond to "True" and "F...

Suppose we randomly pick a testee from this pool ... has the disease. Let T_p be the event that the select... Find the following probabilities. No explanati...

$P(D)=\frac{8}{15}$	Among the 15 testees, 8 have the disease.
$P(T_p \cap D)=\frac{3}{15}=\frac{1}{5}$	Among the 15 testees 3 have the disease an... positive.

In each part below, additional information abo... is given in the condition part. With such infor...

$P(T_p \mid D)=\frac{3}{8}$	Among the 8 testees ... have the disease, 3 have the disease.

Alternatively,

$$P(T_P|D)=\frac{P(T_P\cap D)}{P(D)}=\frac{\frac{1}{5}}{\frac{8}{15}}=\frac{3}{8}.$$

ECS 315: In-Class Exercise # 7 - Sol

Instructions

1. Separate into groups of no more than three students each. **The group cannot be the same as any of your former groups.**
2. Explanation is **not required** for this exercise. [ENRE]
3. **Do not panic.**

Date: 0 5 / 0 9 / 201
Name
Prapun

1) Consider the following sequences of 1s and 0s which summarize the data obtained from 15 testees in a disease testing experiment.

```
D:  0 1 1 0 0 0 0 1 1 1 0 1 0 1 1
TP: 0 0 0 0 1 0 1 1 0 0 1 0 0 1 1
```

The results in the i-th column are for the i-th testee. The D row indicates whether each ... disease under investigation. The TP row indicates whether each of the testees is tested ... Numbers "1" and "0" correspond to "True" and "False", respectively.

Suppose we randomly pick a testee from this pool of 15 testees. Let D be the event t... has the disease. Let T_P be the event that the selected person is tested positive for the ... Find the following probabilities. No explanation is needed here.

There are 15 testees; ... pick one testee; so it ... equal chance of bein... be applied here.

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3 An Introduction to Digital Communication Systems Over Discrete Memoryless Channel

3.1 DMC Models:
Channel Transition Probabilities Q(y|x), Channel Matrix Q, Channel Diagram; Binary Asymmetric Channel

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3 An Introduction to Digital Communication Systems Over Discrete Memoryless Channel

3.1 DMC Models:
MATLAB Simulation

MATLAB

```
%% Generating the channel input x
x = randsrc(1,n,[S_X;p_X]); % channel input

%% Applying the effect of the channel to create the channel output y
y = DMC_Channel_sim(x,S_X,S_Y,Q); % channel output
```

```
function y = DMC_Channel_sim(x,S_X,S_Y,Q)
%% Applying the effect of the channel to create the channel output y
y = zeros(size(x)); % preallocation
for k = 1:length(x)
    % Look at the channel input one by one. Choose the corresponding row
    % from the Q matrix to generate the channel output.
    y(k) = randsrc(1,1,[S_Y;Q(find(S_X == x(k)),:)]);
end
```

[DMC_Channel_sim.m]

[DMC_Analysis_demo.m]

Ex 3.3: BSC

```
%% Simulation parameters
% The number of symbols to be transmitted
n = 20;
% Channel Input
S_X = [0 1]; S_Y = [0 1];
p_X = [0.3 0.7];
% Channel Characteristics
p = 0.1; Q = [1-p p; p 1-p];
```

```
>> BSC_demo
ans =
1 0 1 1 1 1 1 1 1 1 1 1 0 1 0 1 1 1 1 1
ans =
1 1 1 1 1 1 1 1 1 0 1 1 0 1 0 1 1 1 1 1

p_X =
    0.3000    0.7000

Q =
    0.9000    0.1000
    0.1000    0.9000

Elapsed time is 0.134992 seconds.
```

[BSC_demo.m]

Rel. freq. from the simulation

```
%% Statistical Analysis
% The probability values for the channel inputs
p_X                        % Theoretical probability
p_X_sim = hist(x,S_X)/n % Relative frequencies from the simulation
% The probability values for the channel outputs
q = p_X*Q                  % Theoretical probability
q_sim = hist(y,S_Y)/n   % Relative frequencies from the simulation
% The channel transition probabilities from the simulation
Q_sim = [];
for k = 1:length(S_X)
    I = find(x==S_X(k)); LI = length(I);
    rel_freq_Xk = LI/n;
    yc = y(I);
    cond_rel_freq = hist(yc,S_Y)/LI; Q_sim = [Q_sim; cond_rel_freq];
end
Q                          % Theoretical probability
Q_sim                      % Relative frequencies from the simulation
```


[DMC_Analysis_demo.m]

Ex 3.3: BSC

```
%% Simulation parameters
% The number of symbols to be transmitted
n = 20;
% Channel Input
S_X = [0 1]; S_Y = [0 1];
p_X = [0.3 0.7];
% Channel Characteristics
p = 0.1; Q = [1-p p; p 1-p];
```

```
>> BSC_demo
ans =
1 0 1 1 1 1 1 1 1 1 1 1 0 1 0 1 1 1 1 1
ans =
1 1 1 1 1 1 1 1 1 0 1 1 0 1 0 1 1 1 1 1

p_X =
    0.3000    0.7000
p_X_sim =
    0.1500    0.8500
q =
    0.3400    0.6600
q_sim =
    0.1500    0.8500

Q =
    0.9000    0.1000
    0.1000    0.9000
Q_sim =
    0.6667    0.3333
    0.0588    0.9412

Elapsed time is 0.134992 seconds.
```

Because there are only 20 samples, we can't expect the relative freq. from the simulation to match the specified or calculated probabilities.

[BSC_demo.m]

Ex 3.3: BSC

```
%% Simulation parameters
% The number of symbols to be transmitted
n = 1e4;
% Channel Input
S_X = [0 1]; S_Y = [0 1];
p_X = [0.3 0.7];
% Channel Characteristics
p = 0.1; Q = [1-p p; p 1-p];
```

```
>> BSC_demo

p_X =
    0.3000    0.7000
p_X_sim =
    0.3037    0.6963
q =
    0.3400    0.6600
q_sim =
    0.3407    0.6593

Q =
    0.9000    0.1000
    0.1000    0.9000
Q_sim =
    0.9078    0.0922
    0.0934    0.9066

Elapsed time is 0.922728 seconds.
```

[BSC_demo.m]

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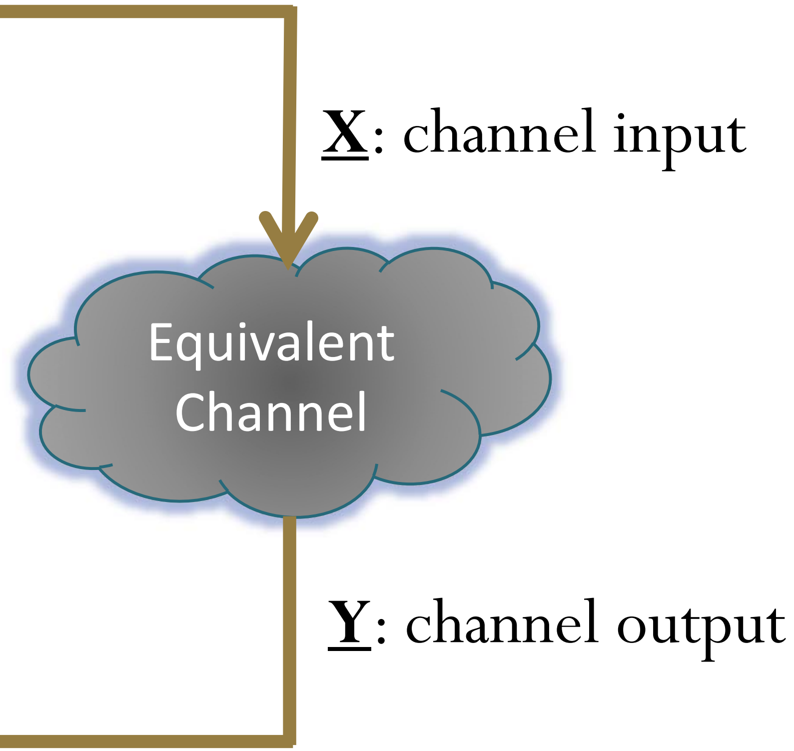
3 An Introduction to Digital Communication Systems Over Discrete Memoryless Channel

3.1 DMC Models:
More Examples on Channel Estimation, Row Sum for the Q matrix

Another Noisy Channel

TTTTTTTTTTTTTTTHTHTTTTTHT
TTHTTTTTTTHTTTTHTTHTTTTTT
TTTTTTTTTTHTTTHTTTHTHTTTT
TTTTHHTTTTTTTTTHTTHTHTTTT



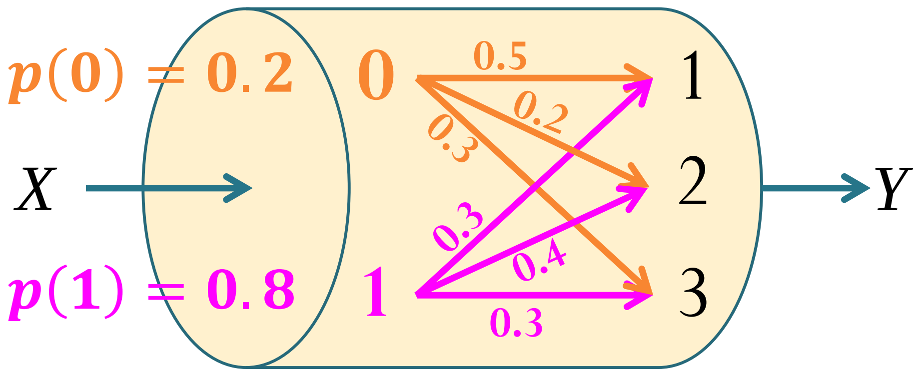
HEHHEHTTTHEETETHETHHHETTH
EEHEEHEHTHHEHEETTHTHEETET
ETEEEEHHTHHETEHEETHTEEHEE
HEHEEHHEHTTTEEHTTTHEHEEHH

[DMC_demo_2.m]

Ex 3.10 & 3.12: DMC

```
%% Simulation parameters
% The number of symbols to be transmitted
n = 20;
% General DMC
% Ex. 3.10 amd 3.12 in lecture note
% Channel Input
S_X = [0 1]; S_Y = [1 2 3];
p_X = [0.2 0.8];
% Channel Characteristics
Q = [0.5 0.2 0.3; 0.3 0.4 0.3];
```

```
>> DMC_demo
ans =
x:  1 1 1 1 1 1 1 1 1 1 0 0 1 1 1 0 1 1 0 1
ans =
y:  1 3 2 2 1 2 1 2 2 3 1 1 1 3 1 3 2 3 1 2
```




```
p_X =
   0.2000   0.8000
p_X_sim =
   0.2000   0.8000
q =
   0.3400   0.3600   0.3000
q_sim =
   0.4000   0.3500   0.2500
Q =
   0.5000   0.2000   0.3000
   0.3000   0.4000   0.3000
Q_sim =
   0.7500        0   0.2500
   0.3125   0.4375   0.2500
>> sym(Q_sim)
ans =
[  3/4,    0, 1/4]
[ 5/16, 7/16, 1/4]
```

[DMC_demo.m]

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3 An Introduction to Digital Communication Systems Over Discrete Memoryless Channel

3.1 DMC Models:

Joint Probability Matrix P, Calculation of Probability of a Statement involving Two Random Variables, Sums along the Rows and Columns of P

Ex 3.10 & 3.13: DMC

```
>> p_X_sim * Q_sim
ans =
    0.4000     0.3500     0.2500
```

Observe that

1. the sum along any row of the **Q** matrix is 1.
2. $\underline{\mathbf{q}} = \underline{\mathbf{p}}\mathbf{Q}$

```
p_X_sim =
   0.2000   0.8000
```

```
q_sim =
   0.4000   0.3500   0.2500
```

```
Q_sim =
   0.7500        0      0.2500
   0.3125   0.4375   0.2500
```

```
>> sym(Q_sim)
ans =
[  3/4,    0, 1/4]
[ 5/16, 7/16, 1/4]
```

Evaluation of Probability from the Joint PMF Matrix

- Consider two random variables X and Y.
- Suppose their **joint pmf matrix** is

$\mathbf{P}_{X,Y}$

x \ y	2	3	4	5	6
1	0.1	0.1	0	0	0
3	0.1	0	0	0.1	0
4	0	0.1	0.2	0	0
6	0	0	0	0	0.3

- Find $P[X + Y < 7]$

Step 1: Find the pairs (x,y) that satisfy the condition "x+y < 7"

One way to do this is to first construct the matrix of $x+y$.

$x + y$

x \ y	2	3	4	5	6
1	3	4	5	6	7
3	5	6	7	8	9
4	6	7	8	9	10
6	8	9	10	11	12

Evaluation of Probability from the Joint PMF Matrix

- Consider two random variables X and Y.
- Suppose their **joint pmf matrix** is

$\mathbf{P}_{X,Y}$

x \ y	2	3	4	5	6
1	0.1	0.1	0	0	0
3	0.1	0	0	0.1	0
4	0	0.1	0.2	0	0
6	0	0	0	0	0.3

- Find $P[X+Y<7]$

Step 2: Add the corresponding probabilities from the joint pmf (matrix)

$x+y$

x \ y	2	3	4	5	6
1	3	4	5	6	7
3	5	6	7	8	9
4	6	7	8	9	10
6	8	9	10	11	12

$$P[X+Y<7] = 0.1+0.1+0.1$$
$$= 0.3$$

Evaluation of Probability from the Joint PMF Matrix

- Consider two random variables X and Y.
- Suppose their **joint pmf matrix** is

$\mathbf{P}_{X,Y}$

x \ y	2	3	4	5	6
1	0.1	0.1	0	0	0
3	0.1	0	0	0.1	0
4	0	0.1	0.2	0	0
6	0	0	0	0	0.3

- Find $P[X = Y]$

$$P[X = Y] = 0 + 0.2 + 0.3 = 0.5$$

Sum of two discrete RVs

- Consider two random variables X and Y.
- Suppose their **joint pmf matrix** is

$\mathbf{P}_{X,Y}$

x \ y	2	3	4	5	6
1	0.1	0.1	0	0	0
3	0.1	0	0	0.1	0
4	0	0.1	0.2	0	0
6	0	0	0	0	0.3

- Find $P[X+Y=7]$

$P[X+Y=7] = 0.1$

$x+y$

x \ y	2	3	4	5	6
1	3	4	5	6	7
3	5	6	7	8	9
4	6	7	8	9	10
6	8	9	10	11	12

Joint to Marginal PMF

- Consider two random variables X and Y.
- Suppose their **joint pmf matrix** is

$$\mathbf{P}_{X,Y} = \begin{array}{c|ccccc} x \backslash y & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0.1 & 0.1 & 0 & 0 & 0 \\ 3 & 0.1 & 0 & 0 & 0.1 & 0 \\ 4 & 0 & 0.1 & 0.2 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0.3 \end{array}$$

- Find $P[X = 3]$

Joint to Marginal PMF

- Consider two random variables X and Y.
- Suppose their **joint pmf matrix** is

$$\mathbf{P}_{X,Y} = \begin{array}{c|ccccc} x \backslash y & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0.1 & 0.1 & 0 & 0 & 0 \\ 3 & 0.1 & 0 & 0 & 0.1 & 0 \\ 4 & 0 & 0.1 & 0.2 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0.3 \end{array}$$

- Find $P[X = 3]$

$$P[X = 3] = 0.1 + 0.1 = 0.2$$

Joint to Marginal PMF

- Consider two random variables X and Y.
- Suppose their **joint pmf matrix** is

$$\mathbf{P}_{X,Y} = \begin{array}{c|ccccc} x \backslash y & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0.1 & 0.1 & 0 & 0 & 0 \\ 3 & 0.1 & 0 & 0 & 0.1 & 0 \\ 4 & 0 & 0.1 & 0.2 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0.3 \end{array}$$

- Find $P[Y = 3]$

Joint to Marginal PMF

- Consider two random variables X and Y.
- Suppose their **joint pmf matrix** is

$$\mathbf{P}_{X,Y} = \begin{array}{c|ccccc} x \backslash y & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0.1 & 0.1 & 0 & 0 & 0 \\ 3 & 0.1 & 0 & 0 & 0.1 & 0 \\ 4 & 0 & 0.1 & 0.2 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0.3 \end{array}$$

- Find $P[Y = 3]$

$$P[Y = 3] = 0.1 + 0.1 = 0.2$$

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3.1 DMC Models:
Calculating Matrix P from Matrix Q, and Vice Versa

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3.1 DMC Models:
Proof of $\underline{q} = \underline{p}Q$;
Remarks on Matrices P and Q

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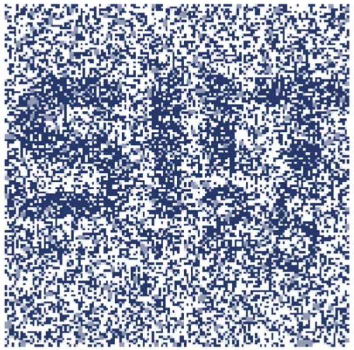
3 An Introduction to Digital Communication Systems Over Discrete Memoryless Channel

3.1 DMC Models:
Examples: Calculating $\underline{q}$ and P from $\underline{p}$ and Q

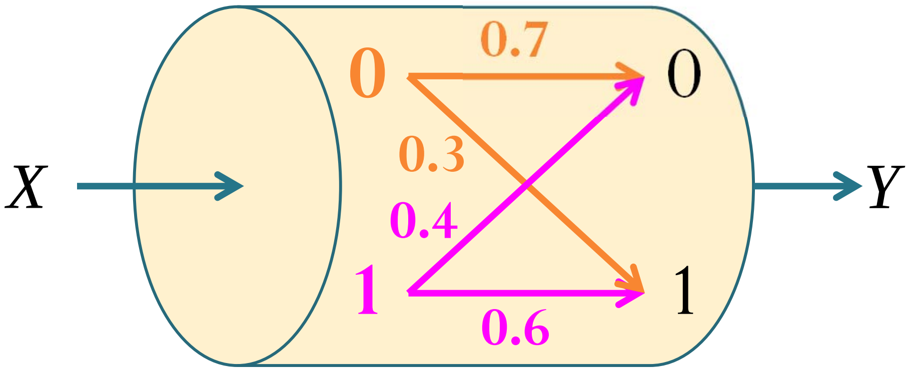
Ex 3.16: BAC

```
S_X = [0 1]; S_Y = [0 1];
p_X = [0.5 0.5];
% Channel Characteristics
Q = [0.7 0.3; 0.4 0.6];
x_raw = imresize(imread('SIIT_25LOGO.png'),[150 150]);
x_raw = [rgb2gray(x_raw) > 128]; % convert to 0 and 1
x = reshape(x_raw,1,n);
```

```
p_X_sim =
   0.2326   0.7674
q =
   0.4698   0.5302
q_sim =
   0.4672   0.5328
Q_sim =
   0.6928   0.3072
   0.3989   0.6011
```



```
>> p = [0.5 0.5];
>> Q = [0.7 0.3; 0.4 0.6];
>> p*Q
ans =
    0.5500    0.4500
>> P = (diag(p))*Q
P =
    0.3500    0.1500
    0.2000    0.3000
>> sum(P)
ans =
    0.5500    0.4500
```



[BAC_demo_image.m]